

# MODULE TENSOR PRODUCT OF SUBNORMAL MODULES NEED NOT BE SUBNORMAL

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**ABSTRACT.** Let  $\kappa : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$  be a diagonal positive definite kernel and let  $\mathcal{H}_\kappa$  denote the associated reproducing kernel Hilbert space of holomorphic functions on the open unit disc  $\mathbb{D}$ . Assume that  $zf \in \mathcal{H}$  whenever  $f \in \mathcal{H}$ . Then  $\mathcal{H}$  is a Hilbert module over the polynomial ring  $\mathbb{C}[z]$  with module action  $p \cdot f \mapsto pf$ . We say that  $\mathcal{H}_\kappa$  is a subnormal Hilbert module if the operator  $\mathcal{M}_z$  of multiplication by the coordinate function  $z$  on  $\mathcal{H}_\kappa$  is subnormal. In [Oper. Theory Adv. Appl, 32: 219-241, 1988], N. Salinas asked whether the module tensor product  $\mathcal{H}_{\kappa_1} \otimes_{\mathbb{C}[z]} \mathcal{H}_{\kappa_2}$  of subnormal Hilbert modules  $\mathcal{H}_{\kappa_1}$  and  $\mathcal{H}_{\kappa_2}$  is again subnormal. In this regard, we describe all subnormal module tensor products  $L_a^2(\mathbb{D}, w_{s_1}) \otimes_{\mathbb{C}[z]} L_a^2(\mathbb{D}, w_{s_2})$ , where  $L_a^2(\mathbb{D}, w_s)$  denotes the weighted Bergman Hilbert module with radial weight

$$w_s(z) = \frac{1}{s\pi} |z|^{\frac{2(1-s)}{s}} \quad (z \in \mathbb{D}, s > 0).$$

In particular, the module tensor product  $L_a^2(\mathbb{D}, w_s) \otimes_{\mathbb{C}[z]} L_a^2(\mathbb{D}, w_s)$  is never subnormal for any  $s \geq 6$ . Thus the answer to this question is no.

## 1. INTRODUCTION

Let  $\mathcal{H}$  be a reproducing kernel Hilbert space of holomorphic functions defined on the unit disc  $\mathbb{D}$  such that  $zf \in \mathcal{H}$  whenever  $f \in \mathcal{H}$ . Thus the linear operator  $\mathcal{M}_z$  of multiplication by the coordinate function  $z$  on  $\mathcal{H}$  is bounded. This allows us to realize  $\mathcal{H}$  as a *Hilbert module* over the polynomial ring  $\mathbb{C}[z]$  with module action given by

$$(p, f) \in \mathbb{C}[z] \times \mathcal{H} \mapsto p(\mathcal{M}_z)f \in \mathcal{H}.$$

Following [9], we say that the Hilbert module  $\mathcal{H}$  is *contractive* if the operator norm of the multiplication operator  $\mathcal{M}_z$  is at most 1. Further, we say that  $\mathcal{H}$  is *subnormal* if  $\mathcal{M}_z$  is subnormal, that is,  $\mathcal{M}_z$  has a normal extension in a Hilbert module containing  $\mathcal{H}$  (refer to [7] for a comprehensive account on subnormal operators). By Agler's Criterion [1, Theorem 3.1],  $\mathcal{H}$  is a contractive subnormal Hilbert module if and only if for every  $f \in \mathcal{H}$ ,  $\phi_f(n) = \|z^n f\|^2$  ( $n \in \mathbb{N}$ ) is completely monotone for every  $f \in \mathcal{H}$ . Recall from [5] that  $\phi : \mathbb{N} \rightarrow (0, \infty)$  is *completely monotone* if

$$\sum_{j=0}^m (-1)^j \binom{m}{j} \phi(n+j) \geq 0 \text{ for all } m, n \in \mathbb{N}.$$

**Remark 1.1 :** Note that if  $\phi$  is a completely monotone sequence then so is  $\psi_m$  for any  $m \in \mathbb{N}$ , where  $\psi_m(n) = \phi(m+n)$  ( $n \in \mathbb{N}$ ).

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We further note that, as a consequence of Hausdorff's solution to the Hausdorff's moment problem [5, Chapter 4, Proposition 6.11],  $\mathcal{H}$  is a contractive subnormal Hilbert module if and only if for every unit vector  $f \in \mathcal{H}$ ,  $\{\|z^n f\|^2\}_{n \in \mathbb{N}}$  is a *Hausdorff moment sequence*, that is, there exists a unique probability measure  $\mu_f$  supported in  $[0, 1]$  such that

$$\|z^n f\|^2 = \int_{[0,1]} t^n d\mu_f \quad (n \in \mathbb{N}).$$

In this text, we are primarily interested in the following one parameter family of subnormal Hilbert modules.

**Example 1.2 :** For a real number  $s > 0$ , consider the Hilbert space  $L_a^2(\mathbb{D}, w_s)$  of holomorphic functions defined on the open unit disc  $\mathbb{D}$  which are square integrable with respect to the weighted area measure  $w_s dA$  with radial weight function

$$w_s(z) = \frac{1}{s\pi} |z|^{\frac{2(1-s)}{s}} \quad (z \in \mathbb{D}).$$

Then  $L_a^2(\mathbb{D}, w_s)$  is a Hilbert module over the polynomial ring  $\mathbb{C}[z]$  (refer to [10] for the basic theory of weighted Bergman spaces). Since  $L_a^2(\mathbb{D}, w_s)$  is a closed subspace of  $L^2(\mathbb{D}, w_s)$ ,  $L_a^2(\mathbb{D}, w_s)$  is a subnormal Hilbert module. Further, since the measure  $w_s dA$  is rotation-invariant, the monomials  $\{z^n\}_{n \in \mathbb{N}}$  are orthogonal in  $L_a^2(\mathbb{D}, w_s)$ . Further,

$$(1.1) \quad \|z^n\|_{L_a^2(\mathbb{D}, w_s)}^2 = \frac{1}{sn + 1} \quad (n \in \mathbb{N}).$$

In particular,  $L_a^2(\mathbb{D}, w_s)$  is a reproducing kernel Hilbert space associated with the diagonal reproducing kernel

$$\kappa_s(z, w) = \frac{s}{(1 - z\bar{w})^2} + \frac{1 - s}{1 - z\bar{w}} \quad (z, w \in \mathbb{D}).$$

**Remark 1.3 :** Note that  $L_a^2(\mathbb{D}, w_1)$  is the unweighted Bergman space  $L_a^2(\mathbb{D})$ . It is worth noting that the definition of  $L_a^2(\mathbb{D}, w_s)$ ,  $s > 0$  extends to the case  $s = 0$ . Indeed,  $L_a^2(\mathbb{D}, w_0)$  may be identified with the Hardy space  $H^2(\mathbb{D})$  of the open unit disc  $\mathbb{D}$ .

In [11], N. Salinas studied the notion of module tensor product of Hilbert  $\mathcal{A}$ -modules for a complex unital algebra  $\mathcal{A}$ . Recall that for Hilbert  $\mathcal{A}$ -modules  $\mathcal{H}$  and  $\mathcal{K}$ ,  $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{K}$  is obtained by tensoring Hilbert modules  $\mathcal{H}$  and  $\mathcal{K}$ , and then dividing out by the natural action of  $\mathcal{A}$  on  $\mathcal{H}$  and  $\mathcal{K}$ . In particular, in [11, Corollary 3.6], it is shown that for the polynomial algebra  $\mathbb{C}[z]$  and for Hilbert modules  $\mathcal{H}_{\kappa_1}$  and  $\mathcal{H}_{\kappa_2}$  associated with so-called sharp, diagonal kernels  $\kappa_1$  and  $\kappa_2$  (of finite rank) respectively, the module tensor product  $\mathcal{H}_{\kappa_1} \otimes_{\mathbb{C}[z]} \mathcal{H}_{\kappa_2}$  is  $\mathbb{C}[z]$ -isomorphic to the Hilbert space  $\mathcal{H}_{\kappa_1 \kappa_2}$  associated with the diagonal kernel  $\kappa_1 \kappa_2$ . In case of scalar valued diagonal kernels  $\kappa_1$  and  $\kappa_2$ , the moment sequences  $\{\|z^n\|_{\mathcal{H}_{\kappa_1}}^2\}_{n \in \mathbb{N}}$ ,  $\{\|z^n\|_{\mathcal{H}_{\kappa_2}}^2\}_{n \in \mathbb{N}}$ ,  $\{\|z^n\|_{\mathcal{H}_{\kappa_1 \kappa_2}}^2\}_{n \in \mathbb{N}}$  are related by the following relations:

$$(1.2) \quad \|z^n\|_{\mathcal{H}_{\kappa_1 \kappa_2}}^2 = \frac{1}{\sum_{k=0}^n \frac{1}{\|z^k\|_{\mathcal{H}_{\kappa_1}}^2} \frac{1}{\|z^{n-k}\|_{\mathcal{H}_{\kappa_2}}^2}} \quad (n \in \mathbb{N}).$$

In [11, Remark 3.7], he asked whether  $\mathcal{H}_{\kappa_1} \otimes_{\mathbb{C}[z]} \mathcal{H}_{\kappa_2}$  is a subnormal Hilbert module for subnormal Hilbert modules  $\mathcal{H}_{\kappa_1}$  and  $\mathcal{H}_{\kappa_2}$  associated with diagonal scalar-valued kernels  $\kappa_1$  and  $\kappa_2$  respectively? In view of (1.2) and the discussion following Remark (1.1), this is equivalent to the following question:

**Question 1.4.** *Whether  $\{\|z^n\|_{\mathcal{H}_{\kappa_1 \kappa_2}}^2\}_{n \in \mathbb{N}}$  is a Hausdorff moment sequence for Hausdorff moment sequences  $\{\|z^n\|_{\mathcal{H}_{\kappa_1}}^2\}_{n \in \mathbb{N}}$  and  $\{\|z^n\|_{\mathcal{H}_{\kappa_2}}^2\}_{n \in \mathbb{N}}$ ?*

The same question in the context of hyponormal Hilbert modules was settled in the affirmative in [4] around the same time (other variations of Question 1.4 have also been investigated, for example, see [3, Proposition 6], [6, Theorem 1.1], and [8, Theorem 7.1]). For subnormal Hilbert modules, it was believed that the answer is no. Indeed, this is true as shown in the following theorem.

**Theorem 1.5.** *Let  $s_1$  and  $s_2$  be positive real numbers and let  $\mathcal{H}_\kappa$  denote the module tensor product  $L_a^2(\mathbb{D}, w_{s_1}) \otimes_{\mathbb{C}[z]} L_a^2(\mathbb{D}, w_{s_2})$  of the weighted Bergman Hilbert modules  $L_a^2(\mathbb{D}, w_{s_1})$  and  $L_a^2(\mathbb{D}, w_{s_2})$ . Then  $1/\|z^n\|_{\mathcal{H}_\kappa}^2$  is a degree 3 polynomial in  $n \in \mathbb{N}$ , say,  $p$ . If  $\tilde{p}$  denotes the analytic extension of  $p$  to the complex plane, then we have the following:*

- (1) *if  $\tilde{p}$  has all real roots then  $\mathcal{H}_\kappa$  is a subnormal Hilbert module if and only if all roots of  $\tilde{p}$  lie in  $\mathbb{L}_0$ ;*
- (2) *if  $\tilde{p}$  has a non-real complex root  $z_0$  then  $\mathcal{H}_\kappa$  is a subnormal Hilbert module if and only if  $z_0$  lies in the closure of  $\mathbb{L}_{-1}$ .*

where, for a real number  $r$ ,  $\mathbb{L}_r$  denotes the open left half plane

$$\mathbb{L}_r := \{z \in \mathbb{C} : \text{real part of } z \text{ is less than } r\}.$$

**Corollary 1.6.** *Let  $s_1$  and  $s_2$  be positive real numbers and let  $\mathcal{H}_\kappa$  denote the module tensor product  $L_a^2(\mathbb{D}, w_{s_1}) \otimes_{\mathbb{C}[z]} L_a^2(\mathbb{D}, w_{s_2})$  of the weighted Bergman Hilbert modules  $L_a^2(\mathbb{D}, w_{s_1})$  and  $L_a^2(\mathbb{D}, w_{s_2})$ . Let  $\mathfrak{s}$  and  $\mathfrak{p}$  denote the sum and product of  $s_1, s_2$  respectively. Then we have the following:*

- (1) *If  $(3\mathfrak{s} - \mathfrak{p})^2 \geq 24\mathfrak{p}$  then  $\mathcal{H}_\kappa$  is subnormal if and only if  $3\mathfrak{s} > \mathfrak{p}$ .*
- (2) *If  $(3\mathfrak{s} - \mathfrak{p})^2 < 24\mathfrak{p}$  then  $\mathcal{H}_\kappa$  is a subnormal Hilbert module if and only if  $\mathfrak{s} \geq \mathfrak{p}$ .*

**Remark 1.7 :** If  $\mathfrak{s} \geq \mathfrak{p}$  then  $L_a^2(\mathbb{D}, w_{s_1}) \otimes_{\mathbb{C}[z]} L_a^2(\mathbb{D}, w_{s_2})$  is always subnormal. Also, if  $3\mathfrak{s} \leq \mathfrak{p}$  then  $L_a^2(\mathbb{D}, w_{s_1}) \otimes_{\mathbb{C}[z]} L_a^2(\mathbb{D}, w_{s_2})$  is never subnormal.

The following is immediate from the preceding remark.

**Corollary 1.8.** *For a positive number  $s$ , let  $\mathcal{H}_\kappa$  denote the module tensor product  $L_a^2(\mathbb{D}, w_s) \otimes_{\mathbb{C}[z]} L_a^2(\mathbb{D}, w_s)$  of the weighted Bergman Hilbert module  $L_a^2(\mathbb{D}, w_s)$  with itself. Then we have the following:*

- (1) *If  $s \leq 2$  then  $\mathcal{H}_\kappa$  is always subnormal.*
- (2) *If  $s \geq 6$  then  $\mathcal{H}_\kappa$  never subnormal.*

The proofs of Theorem 1.5 and Corollary 1.6 will be presented in the next section. Let us illustrate these results with the help of some instructive examples of different flavor.

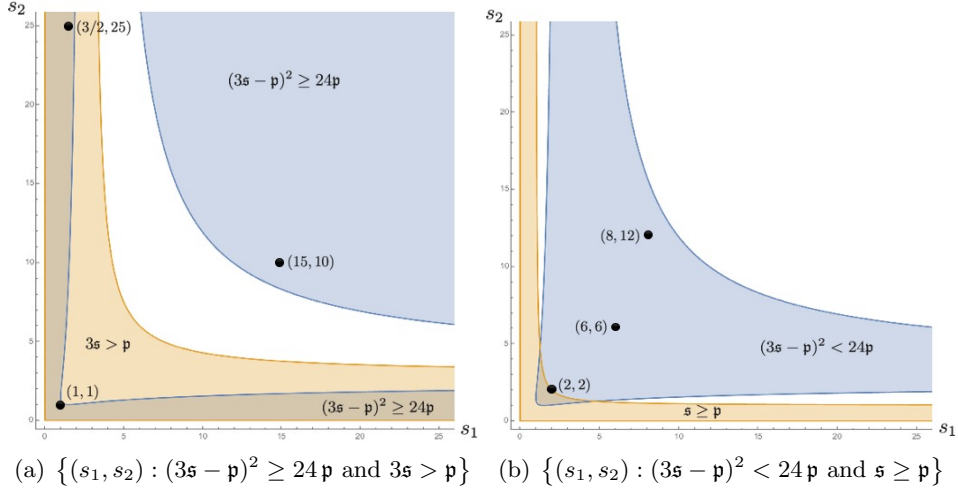


FIGURE 1. Regions of subnormality of the module tensor product  $L_a^2(\mathbb{D}, w_{s_1}) \otimes_{\mathbb{C}[z]} L_a^2(\mathbb{D}, w_{s_2})$  in  $s_1$ - $s_2$  plane, where  $\mathfrak{s}$  and  $\mathfrak{p}$  denote the sum and product of  $s_1, s_2$  respectively; specific cases investigated in Example 1.9 have been shown as points in  $s_1$ - $s_2$  plane.

**Example 1.9 :** Let  $\mathcal{H}_\kappa, \tilde{p}$  be as given in Theorem 1.5 and let

$$\mathcal{D}_m = \sum_{j=0}^m (-1)^j \binom{m}{j} \|z^j\|_{\mathcal{H}_\kappa}^2 \quad (m \in \mathbb{N}).$$

Let  $\mathfrak{s}$  and  $\mathfrak{p}$  denote the sum and product of  $s_1, s_2$  respectively. Then we have the following:

- (1) The case in which  $\tilde{p}$  has real roots, that is,  $(3\mathfrak{s} - \mathfrak{p})^2 > 24\mathfrak{p}$  (see Figure 1(a)):
  - (a) If  $s_1 = 15, s_2 = 10$  then  $3\mathfrak{s} < \mathfrak{p}$ . In this case, the roots of  $\tilde{p}$  are  $-1, 1/10, 2/5$  and  $\mathcal{D}_{75} < 0$ .
  - (b) If  $s_1 = 1, s_2 = 1$  then  $\mathfrak{s} > \mathfrak{p}$ . In this case, the roots of  $\tilde{p}$  are  $-1, -2, -3$  and  $\mathcal{D}_m \geq 0$  for all  $m \in \mathbb{N}$ . The later part may be concluded from Corollary 1.6(1).
  - (c) If  $s_1 = 3/2, s_2 = 25$  then  $3\mathfrak{s} > \mathfrak{p}$ . In this case, roots are  $-1, 2(-7 \pm 2\sqrt{6})/25$  and  $\mathcal{D}_m \geq 0$  for all  $m \in \mathbb{N}$ . Once again, the later part may be concluded from Corollary 1.6(1).
- (2) The case in which  $\tilde{p}$  has a complex root, that is,  $(3\mathfrak{s} - \mathfrak{p})^2 < 24\mathfrak{p}$  (see Figure 1(b)):
  - (a) If  $s_1 = 6, s_2 = 6$ , then  $\mathfrak{s} < \mathfrak{p}$ . In this case, the roots of  $\tilde{p}$  are  $-1, \pm i/\sqrt{6}$  and  $\mathcal{D}_{73} < 0$ .
  - (b) If  $s_1 = 8, s_2 = 12$ , then  $\mathfrak{s} < \mathfrak{p}$ . In this case, the roots of  $\tilde{p}$  are  $-1, (3 \pm i\sqrt{7})/16$  and  $\mathcal{D}_{73} < 0$ .
  - (c) If  $s_1 = 2, s_2 = 2$  then  $\mathfrak{s} = \mathfrak{p}$ . In this case, the roots of  $\tilde{p}$  are  $-1, -1 \pm i/\sqrt{2}$  and  $\mathcal{D}_m \geq 0$  for all  $m \in \mathbb{N}$ . The later part may be concluded from Corollary 1.6(2).

In parts (1)(a) and (2)(a) of Example 1.9, for sufficiently large values of  $m$ , the multiplication operator  $\mathcal{M}_z$  on  $\mathcal{H}_\kappa$  is not  $m$ -hypercontractive in the sense of J. Agler [1]. Moreover, in these examples,  $m = 75, 74$  are the smallest integers for which  $m$ -hypercontractivity of  $\mathcal{M}_z$  fails. It also indicates that direct verification of the non-subnormality of  $\mathcal{M}_z$  is tedious.

## 2. PROOF OF THEOREM 1.5

Recall that, for a real number  $r$ ,  $\mathbb{L}_r$  denotes the open left half plane

$$\mathbb{L}_r = \{z \in \mathbb{C} : \text{real part of } z \text{ is less than } r\}.$$

In the proof of Theorem 1.5, we need the following lemma.

**Lemma 2.1.** *If  $a_0 \in (-\infty, 0)$  and  $a_1, a_2 \in \mathbb{L}_0$ , then*

$$\frac{1}{(n - a_0)(n - a_1)(n - a_2)} = \int_{[0,1]} t^n w(t) dt \quad (n \in \mathbb{N}),$$

where the weight function  $w : [0, 1] \rightarrow (0, \infty)$  is given as follows:

(1) *If  $a_0, a_1, a_2$  are distinct real numbers, then*

$$w(t) = \frac{t^{-a_0-1}}{(a_0 - a_1)(a_0 - a_2)} + \frac{t^{-a_1-1}}{(a_1 - a_0)(a_1 - a_2)} + \frac{t^{-a_2-1}}{(a_2 - a_0)(a_2 - a_1)} \quad (t \neq 0).$$

(2) *If  $a_1 = a_0$  and  $a_2$  is a real number not equal to  $a_0$ , then*

$$w(t) = \frac{1}{(a_0 - a_2)^2} (t^{-a_2-1} - t^{-a_0-1}) - \frac{1}{a_0 - a_2} t^{-a_0-1} \log t \quad (t \neq 0).$$

(3) *If  $a_0 = a_1 = a_2$ , then*

$$w(t) = \frac{1}{2} t^{-a_0-1} (\log t)^2 \quad (t \neq 0).$$

(4) *If  $a_1 = a + ib$ ,  $a_2 = a - ib$  are complex numbers with  $b > 0$  then*

$$w(t) = \frac{1}{(a_0 - a)^2 + b^2} \left( t^{-a_0-1} - \frac{\sqrt{(a_0 - a)^2 + b^2}}{b} t^{-a-1} \sin(b \log t + \theta) \right) \quad (t \neq 0),$$

where  $\theta$  denotes the principal argument of  $a_1 - a_0$ .

*Proof.* The first three parts are special cases of [2, Theorem 3.1]. To see the last part, note that

$$\begin{aligned} & \frac{1}{(n - a_0)(n - a_1)(n - a_2)} \\ &= \frac{1}{(a - a_0)^2 + b^2} \left( \frac{1}{n - a_0} + \frac{a_2 - a_0}{2ib} \frac{1}{n - a_1} - \frac{a_1 - a_0}{2ib} \frac{1}{n - a_2} \right). \end{aligned}$$

Further, the term in brackets can be rewritten as

$$\int_0^1 t^{n-a_0-1} dt + \frac{a_2 - a_0}{2ib} \int_0^1 t^{n-a_1-1} dt - \frac{a_1 - a_0}{2ib} \int_0^1 t^{n-a_2-1} dt.$$

However,

$$\begin{aligned} & \frac{a_2 - a_0}{2ib} t^{-a_1-1} - \frac{a_1 - a_0}{2ib} t^{-a_2-1} = \\ & - \frac{\sqrt{(a - a_0)^2 + b^2}}{b} \left( \frac{e^{i(b \log t + \theta)} - e^{-i(b \log t + \theta)}}{2i} \right) t^{-a-1}, \end{aligned}$$

where  $\theta$  denotes the principal argument of  $a_1 - a_0$ . It is now easy to see that  $w(t)$  has the desired form.  $\square$

**Remark 2.2 :** Note that in all the above cases,  $\int_{[0,1]} w(t)dt \in (0, \infty)$ .

*Proof of Theorem 1.5.* We have already recorded in (1.1) that

$$\|z^n\|_{L_a^2(\mathbb{D}, w_{s_j})}^2 = \frac{1}{s_j n + 1} \text{ for } n \in \mathbb{N} \text{ and } j = 1, 2.$$

It now follows from (1.2) that

$$\|z^n\|_{\mathcal{H}_\kappa}^2 = \frac{1}{\sum_{k=0}^n (s_1 k + 1)(s_2(n - k) + 1)} \quad (n \in \mathbb{N}).$$

However,

$$\begin{aligned} \sum_{k=0}^n (s_1 k + 1)(s_2(n - k) + 1) &= \frac{1}{6}(n+1)(s_1 s_2 n^2 + (3(s_1 + s_2) - s_1 s_2)n + 6) \\ (2.3) \qquad \qquad \qquad &= \frac{1}{\alpha_1 \alpha_2}(n+1)(n - \alpha_1)(n - \alpha_2), \end{aligned}$$

where  $\alpha_1, \alpha_2$  are given by

$$(2.4) \quad \alpha_1 := \frac{-\gamma + \sqrt{\gamma^2 - 24s_1 s_2}}{2s_1 s_2}, \quad \alpha_2 := \frac{-\gamma - \sqrt{\gamma^2 - 24s_1 s_2}}{2s_1 s_2}$$

with  $\gamma := 3(s_1 + s_2) - s_1 s_2$ . It follows that

$$\|z^n\|_{\mathcal{H}_\kappa}^2 = \frac{\alpha_1 \alpha_2}{(n+1)(n - \alpha_1)(n - \alpha_2)} \quad (n \in \mathbb{N}).$$

This shows that  $1/\|z^n\|_{\mathcal{H}_\kappa}^2$  is a degree 3 polynomial  $p$  in  $n \in \mathbb{N}$ .

Let  $\tilde{p}$  denote the analytic extension of  $p$  to the complex plane. Assume now that one of  $\alpha_1$  and  $\alpha_2$  belongs to  $\mathbb{C} \setminus \mathbb{L}_0$ . By (2.3) and (2.4), however, both  $\alpha_1$  and  $\alpha_2$  must belong to  $\mathbb{C} \setminus \mathbb{L}_0$ . We contend that the sequence

$$\left\{ \|z^n\|_{\mathcal{H}_\kappa}^2 = \frac{\alpha_1 \alpha_2}{(n+1)(n - \alpha_1)(n - \alpha_2)} \right\}_{n \in \mathbb{N}}$$

is not a Hausdorff moment sequence. On the contrary, suppose that there exists a probability measure  $\mu$  on  $[0, 1]$  such that

$$\|z^n\|_{\mathcal{H}_\kappa}^2 = \int_{[0,1]} t^n d\mu \quad (n \in \mathbb{N}).$$

Let  $n_0$  be the smallest integer bigger than the maximum of real parts of  $\alpha_1$  and  $\alpha_2$ . Consider now the completely monotone sequence  $\{\|z^{n+n_0}\|_{\mathcal{H}_\kappa}^2\}_{n \in \mathbb{N}}$  (Remark 1.1), and note that by the preceding lemma,

$$\|z^{n+n_0}\|_{\mathcal{H}_\kappa}^2 = \frac{\alpha_1 \alpha_2}{(n + n_0 + 1)(n + n_0 - \alpha_1)(n + n_0 - \alpha_2)} = \int_{[0,1]} t^{n+n_0} w(t) dt$$

for some non-negative integrable function  $w : [0, 1] \rightarrow (0, \infty)$ . On the other hand,  $\|z^{n+n_0}\|_{\mathcal{H}_\kappa}^2 = \int_{[0,1]} t^{n+n_0} d\mu(t)$ . By the determinacy of the Hausdorff moment problem [5], we must have

$$t^{n_0} d\mu(t) = w(t) dt.$$

We now apply the previous lemma to  $a_0 := -n_0 - 1$ ,  $a_1 := \alpha_1 - n_0$  and  $a_2 := \alpha_2 - n_0$  to conclude that  $t^{-n_0}w(t)$  takes one of the following expressions (for some scalars  $c_0, c_1, c_2, \theta$ ):

(1) If  $a_1, a_2$  are distinct real numbers, then

$$t^{-n_0}w(t) = c_0 + c_1 t^{-\alpha_1-1} + c_2 t^{-\alpha_2-1}.$$

(2) If  $a_1 = a_2$ , then

$$t^{-n_0}w(t) = c_0(1 - t^{-\alpha_1-1}) + c_1 t^{-\alpha_1-1} \log t.$$

(3) If  $a_1 = c + ib$ ,  $a_2 = c - ib$  are complex numbers with  $b > 0$  then

$$t^{-n_0}w(t) = c_0 t^{-a_0-1} + c_1 t^{-\alpha-1} \sin(b \log t + \theta),$$

where  $\theta$  is the principal argument of  $a_1 - a_0$ .

One way to get a contradiction is to show that the integral of  $t^{-n_0}w(t)$  over  $[0, 1]$  is not convergent. However, to avoid computations, one can alternatively arrive at the contradiction through an interpolation result on completely monotone sequences [12]. Toward this, observe that  $\{\|z^n\|_{\mathcal{H}_\kappa}^2\}_{n \in \mathbb{N}}$  is minimal in the sense that  $\mu(\{0\}) = 0$ , where we used the convention that  $\infty \cdot 0 = 0$ . By [2, Proposition 4.1], all the zeros of  $\tilde{p}(z) = (z+1)(z-\alpha_1)(z-\alpha_2)$  must lie in  $\mathbb{L}_0$ , which contradicts the assumption that at least one of  $\alpha_1$  and  $\alpha_2$  belongs to  $\mathbb{C} \setminus \mathbb{L}_0$ . The remaining part in (1) is immediate from [2, Theorem 3.1].

To see (2), assume that  $\tilde{p}$  has a non-real complex root  $\alpha_1 = a + ib \in \mathbb{L}_0$  with  $b > 0$ . In view of the preceding discussion, it suffices to show that  $\mathcal{H}_\kappa$  is a subnormal Hilbert module if and only if  $\alpha_1$  lies in the closure of  $\mathbb{L}_{-1}$ . By (4) of the preceding lemma,

$$\begin{aligned} w(t) &= \frac{\alpha_1 \alpha_2}{(1+a)^2 + b^2} \left( 1 - \frac{\sqrt{(1+a)^2 + b^2}}{b} t^{-a-1} \sin(b \log t + \theta) \right) \\ (2.5) \quad &= \frac{|\alpha_1|^2}{(1+a)^2 + b^2} \left( t^{a+1} - \frac{\sin(b \log t + \theta)}{\sin \theta} \right) t^{-a-1}, \end{aligned}$$

where  $\theta$  denotes the principal argument of  $a + 1 + ib$ . If  $a = -1$  then  $w(t) = \frac{|\alpha_1|^2}{b^2} (1 - \sin(b \log t + \theta)) \geq 0$  for  $t \in (0, 1]$ , and hence  $\|z^n\|_{\mathcal{H}_\kappa}^2$  is a Hausdorff moment sequence. Assume now that  $\alpha_1 = a + ib$  lies in  $\mathbb{L}_{-1}$ , that is,  $a < -1$ . Note that  $\pi/2 < \theta < \pi$ . Consider the function

$$F(t) := t^{a+1} - \frac{\sin(b \log t + \theta)}{\sin \theta} \quad (t > 0),$$

and note that  $F'(t) \leq \frac{1}{t} \left( a + 1 - \frac{b \cos(b \log t + \theta)}{\sin \theta} \right)$ . Let  $t_0 := e^{-2\theta/b}$ . Then, for  $t \in [t_0, 1]$ , we have

$$F'(t) \leq \frac{1}{t} \left( a + 1 - \frac{b \cos \theta}{\sin \theta} \right) = 0.$$

Hence  $F(t) \geq F(1) = 0$  for all  $t \in [t_0, 1]$ . For  $t \in (0, t_0)$ , we have

$$\begin{aligned} F(t) &\geq e^{-2\theta(a+1)/b} - \frac{1}{\sin \theta} \\ &\geq e^{-2(a+1)/b} - \sqrt{1 + \left(\frac{a+1}{b}\right)^2} \\ &\geq e^{-2(a+1)/b} - \left(1 + \frac{1}{2}\left(\frac{a+1}{b}\right)^2\right), \end{aligned}$$

which is clearly positive. It is now clear from (2.5) that  $\|z^n\|_{\mathcal{H}_\kappa}^2$  is a Hausdorff moment sequence.

Assume next that  $-1 < a < 0$ . Then  $0 < \theta < \pi/2$ . Choose  $m \in \mathbb{Z}$  such that  $t_m^{a+1} < \frac{1}{2\sin \theta}$ , where  $t_m := e^{(2m\pi + \pi/2 - \theta)/b}$ . Then, by (2.5),

$$\begin{aligned} w(t_m) &= \frac{|\alpha_1|^2}{(1+a)^2 + b^2} \left(t_m^{a+1} - \frac{1}{\sin \theta}\right) t_m^{-a-1} \\ &\leq -\frac{|\alpha_1|^2}{(1+a)^2 + b^2} \frac{t_m^{-a-1}}{2\sin \theta} < 0. \end{aligned}$$

By the continuity of  $w$ ,  $w < 0$  in a neighborhood of  $t_m$ . It follows that  $\|z^n\|_{\mathcal{H}_\kappa}^2$  is not a Hausdorff moment sequence. This completes the proof.  $\square$

Finally, we note that the conclusions in Corollary 1.6 follow from Theorem 1.5 and (2.4).

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